

Predictive control for constrained discrete-time periodic systems using a time-varying terminal region ^{*}

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Abstract: To guarantee stability of a model predictive control scheme it is essential to suitably calculate the terminal region and the terminal penalty term. In this paper we propose an approach to overcome this problem for the class of periodically time-varying systems. We consider both systems with periodic linear dynamics as well as systems with periodic nonlinear dynamics where the nonlinearities can be approximated with polytopic linear differential inclusions. In both cases exploiting the periodicity of the system dynamics leads to linear matrix inequality (LMI) conditions which can be used to calculate the terminal region and the terminal penalty term. The LMI conditions are shown to be less conservative than existing approaches applicable to the considered system class.

Keywords: Predictive control, periodic systems, linear matrix inequalities

1. INTRODUCTION

Systems with periodically time-varying dynamics are of great importance for engineering applications. Examples of processes that can be modeled through a periodic system are sampled-data systems, satellites (Psiaki [2001]), rotors of helicopters (Arcara et al. [2000]), or chemical processes. Several methods to guarantee stability of linear periodic systems have been developed in the past, see e.g. Bittanti et al. [1984], Bolzern and Colaneri [1988], De Souza and Trofino [2000], Farges et al. [2005, 2007]. A survey on the analysis and control of periodic systems is given in Bittanti and Colaneri [1999, 2009]. So far, only few model predictive control (MPC) schemes have been developed for periodically time-varying systems with linear dynamics (in the following also referred to as linear periodic systems), see e.g. De Nicolao [1994], Kim et al. [2000], Kwon and Byun [1989], Böhm et al. [2009]. The goal of this paper is to derive an MPC controller for the class of state and input constrained discrete-time periodic systems. We consider both systems with periodically time-varying linear dynamics as well as systems with periodic nonlinear dynamics, where the nonlinearities can be approximated using polytopic linear differential inclusion (PLDI, Boyd et al. [1994]) techniques.

The basic idea of model predictive control is as follows: By solving online a finite horizon open-loop optimal control problem based on current measurements of the system state, an optimal input trajectory is obtained. The first part of this trajectory is applied to the system and the optimal control problem is solved again based on new

measurements at the next sampling instant. Although MPC often leads to good controller performance, closed-loop stability is not naturally guaranteed. Several MPC schemes use a terminal region and a terminal penalty term, both calculated offline, to guarantee closed-loop stability, see e.g. Chen and Allgöwer [1998], Mayne et al. [2000], Fontes [2000]. However, the calculation of the terminal region and the terminal penalty term is generally not a trivial task, see e.g. Chen and Ballance [1999], Chen and Allgöwer [1998], Böhm et al. [2008], Yu et al. [2009], and one often has to exploit the structure of the system class considered, as e.g. the periodicity of the system dynamics, in order to obtain reasonable results. In principle, the approaches presented in Lee et al. [2005] and Yu et al. [2009] apply to periodic systems if the time-varying system matrices are considered as extreme matrices of the required PLDI formulation. However, they do not explicitly take the periodic structure of the considered system class into account and therefore the obtained LMI conditions suffer from conservativeness, which might lead to non-desirable small terminal regions or even infeasibility. To overcome this problem in this paper we derive LMI conditions for the calculation of periodically time-varying terminal regions and terminal penalty terms using the periodicity of the system dynamics. Furthermore, we extend the result to the case of nonlinear periodic systems which can be approximated using a PLDI formulation. It is shown that the obtained conditions are less conservative than those of Lee et al. [2005] and Yu et al. [2009], thus leading to an improved size of the terminal region. The approach presented in this paper uses the ideas of Böhm et al. [2009], and Reble et al. [2009] for the nonlinear case respectively, where a convex optimization problem based on LMIs is solved repeatedly online to obtain a time-varying, stabi-

^{*} The authors gratefully acknowledge funding by the German Research Foundation (AL 316/5-1) and support by the Graduate School of the Stuttgart Research Centre for Simulation Technology (SimTech).

lizing feedback law for periodic systems.

The remainder of the paper is structured as follows: Section 2 introduces the system class and the MPC control problem considered. Section 3 derives the main result of this paper, namely LMI conditions for the calculation of periodically time-varying terminal regions and terminal penalty terms for the class of linear periodic systems. These results are extended in Section 4 to the nonlinear case. Section 5 illustrates the results obtained via a simulation example. The paper is concluded in Section 6 with a brief summary.

2. PROBLEM SETUP

Consider the N -periodic discrete-time system

$$x_{k+1} = A_k x_k + B_k u_k \quad (1)$$

with initial condition $x_0 = \bar{x}_0$. In (1) $x_k \in \mathbb{R}^n$ is the system state, $u_k \in \mathbb{R}^m$ the control input, $k \geq 0$ the time variable, and $A_{k+N} = A_k \in \mathbb{R}^{n \times n}$, $B_{k+N} = B_k \in \mathbb{R}^{n \times m}$ are linear periodic matrices with the time period N .

The control problem is to stabilize the origin of system (1) such that the state and input constraints defined by the polyhedral set

$$\mathcal{C} := \left\{ \begin{bmatrix} x_k \\ u_k \end{bmatrix} \in \mathbb{R}^{n+m} : c_j x_k + d_j u_k \leq 1, j = 1, \dots, p \right\} \quad (2)$$

are satisfied at every time instant k , where $c_j \in \mathbb{R}^{1 \times n}$ and $d_j \in \mathbb{R}^{1 \times m}$. Since MPC takes constraints explicitly into account it is a suitable choice to achieve the given control task. The basic idea of MPC is to repeatedly solve online at each time instant k an open-loop optimal control problem based on the measured system state x_k . Here, we consider the finite horizon quadratic cost function

$$J(x_k, k, \mathbf{u}_k) := \sum_{i=0}^{H-1} x_{k+i|k}^T Q x_{k+i|k} + u_{k+i|k}^T R u_{k+i|k} + x_{k+H|k}^T P_{k+H} x_{k+H|k}, \quad (3)$$

with the positive definite weighting matrices $Q \in \mathbb{R}^{n \times n}$ and $R \in \mathbb{R}^{m \times m}$, and the positive definite terminal penalty matrix $P_{k+H} \in \mathbb{R}^{n \times n}$. The sequence $\mathbf{u}_k = [u_{k|k}, u_{k+1|k}, \dots, u_{k+H-1|k}]$ is the control input trajectory and H denotes the prediction horizon. Thus, the optimal control problem inherent to the MPC controller is

$$\underset{\mathbf{u}_k}{\text{minimize}} J(x_k, k, \mathbf{u}_k) \quad (4a)$$

subject to

$$x_{k+i+1|k} = A_{k+i} x_{k+i|k} + B_{k+i} u_{k+i|k}, x_{k|k} = x_k, \quad (4b)$$

$$\begin{bmatrix} x_{k+i+1|k} & u_{k+i|k} \end{bmatrix}^T \in \mathcal{C}, i = 0, \dots, H-1, \quad (4c)$$

$$x_{k+H|k} \in \mathcal{E}_{k+H} = \{y \in \mathbb{R}^n : y^T P_{k+H} y \leq \alpha\}. \quad (4d)$$

In (4) the index $k+i|k$ denotes the predicted states and inputs at the predicted time instant $k+i$, where the prediction has been calculated at the current time instant k . This notation is necessary since the trajectories predicted at time k in general differ from those predicted at time $k+1$.

The solution to the optimization problem (4) at time k is the open-loop input trajectory

$$\mathbf{u}_k^* = \arg \min_{\mathbf{u}_k} J(x_k, k, \mathbf{u}_k) \quad (5)$$

We denote the optimal value of the cost function by

$$J^*(x_k, k) := J(x_k, k, \mathbf{u}_k^*). \quad (6)$$

The control law applied to system (1) is updated at each time instant k by the repeated solution of the optimization problem (4), i.e. the applied input is

$$u_k = u_{k|k}^*, \quad (7)$$

where $u_{k|k}^*$ is the first part of the optimal input sequence \mathbf{u}_k^* .

In the optimization problem above the state at the end of the prediction horizon $x_{k+H|k}$ is penalized in the cost function (3) by the terminal penalty term $x_{k+H|k}^T P_{k+H} x_{k+H|k}$. Furthermore, it is forced to lie in the terminal region \mathcal{E}_{k+H} , which is an ellipsoid defined by the matrix P_{k+H} and the scalar $\alpha \in \mathbb{R}^+$. Note that the terminal penalty matrix P_{k+H} and the terminal region \mathcal{E}_{k+H} are chosen to be *time-varying*, in contrast to standard predictive control schemes, see e.g. Chen and Allgöwer [1998], Fontes [2000], Mayne et al. [2000], Chen and Ballance [1999], Yu et al. [2009], Böhm et al. [2008]. If chosen suitably, the time-varying matrix P_{k+H} and the scalar α , which define the terminal penalty term and the terminal region, guarantee closed-loop stability of the considered MPC scheme. The basic idea of this paper is to exploit the periodic structure of the system class considered to calculate N matrices P_0, \dots, P_{N-1} via the solution of linear matrix inequalities such that closed-loop stability can be established. In the open-loop optimal control problem (4) those matrices are applied periodically, i.e.

$$P_{k+N} = P_k \quad \forall k, \quad (8)$$

and consequently

$$\mathcal{E}_{k+N} = \mathcal{E}_k \quad \forall k. \quad (9)$$

In the following section we derive LMI conditions for the calculation of the terminal weighting matrices (and therefore of the time-varying terminal region). It will be shown that the obtained LMIs are less conservative than those of comparable approaches, as e.g. Lee et al. [2005], Yu et al. [2009], thus leading to improved feasibility properties and a larger terminal region.

3. MAIN RESULT

The calculation of a time-varying terminal region requires the simultaneous calculation of a time-varying terminal controller, which in this paper is chosen to be

$$\tilde{u}_k = K_k x_k, \quad (10)$$

where $K_{k+N} = K_k \in \mathbb{R}^{m \times n}$ is a periodically time-varying feedback matrix. Note that the terminal control law is never applied to the system, however its existence is required in the MPC stability proof. The considered terminal control law (10) renders the input state dependent. Therefore, the constraint set (2) translates into a periodically time-varying set depending only on the system state x_k

$$\mathcal{C}_k := \left\{ x_k \in \mathbb{R}^n : (c_j + d_j K_k) x_k \leq 1, j = 1, \dots, p \right\}. \quad (11)$$

Thus, if at a specific time instant k the state x_k lies in the set \mathcal{C}_k , then both state and input constraints are satisfied at this time instant. Since the terminal region (4d) is

defined as a periodically time-varying ellipsoid \mathcal{E}_k , clearly state and input constraints are satisfied if the considered ellipsoid lies in the constraint set \mathcal{C}_k at the corresponding time instant. The following lemma provides necessary and sufficient conditions for the ellipsoid \mathcal{E}_k lying in the polytope \mathcal{C}_k .

Lemma 1. The ellipsoid $\mathcal{E}_k := \{y \in \mathbb{R}^n : y^T P_k y \leq \alpha\}$ lies in the constraint set \mathcal{C}_k at time instant k if and only if

$$(c_j + d_j K_k) \alpha P_k^{-1} (c_j + d_j K_k)^T \leq 1, \quad \forall j = 1, \dots, p. \quad (12)$$

Proof 1. The proof can be found in Boyd et al. [1994] and in Chen and Ballance [1999].

Furthermore, the MPC stability proof will require the (time-varying) optimal value function $J^*(x_k, k)$ to be positive definite. This property is established by the following lemma.

Lemma 2. The optimal value function $J^*(x, k)$ is positive definite for all $x \in D$, where D is the feasible region of the optimization problem (4) at the initial time instant $k = 0$.

Proof 2. To show positive definiteness of the optimal value function we have to show that

- a.) $J^*(0, k) = 0 \quad \forall k \geq 0$,
- b.) \exists a time-invariant positive definite function $V(x)$ such that

$$V(x) \leq J^*(x, k) \quad \forall x \in D, \quad \forall k \geq 0, \quad (13)$$

see Marquez [2003]. It trivially follows from the definition of the cost function (3) that property a.) holds. Furthermore, we know from (3) that

$$x^T Q x = V(x) \leq J^*(x, k) \quad \forall x \in D, \quad \forall k \geq 0. \quad (14)$$

Per definition of the set D the optimal value function $J^*(x, 0)$ exists for all $x \in D$. This concludes the proof under the assumption that $J^*(x, k)$ exists for all $k > 0$ if it exists for $k = 0$, i.e. the optimal control problem (4) is feasible at all time instants $k > 0$ if it is initially feasible.

Using the results derived in Lemma 1 and Lemma 2, the following theorem provides LMI conditions which can be used to calculate the terminal region, the terminal penalty term and the terminal controller such that the MPC scheme defined by (3)-(7) is asymptotically stable.

Theorem 1. Suppose that the open-loop optimal control problem (4) has a feasible solution at initial time $k = 0$, and that there exist matrices $0 < X_k = X_k^T \in \mathbb{R}^{n \times n}$ and $Y_k \in \mathbb{R}^{m \times n}$, and a constant $\alpha \in \mathbb{R}^+$ such that the LMIs

$$\begin{bmatrix} X_k & X_k A_k^T + Y_k^T B_k^T & X_k Q^{\frac{1}{2}} Y_k^T R^{\frac{1}{2}} \\ A_k X_k + B_k Y_k & X_{k+1} & 0 & 0 \\ Q^{\frac{1}{2}} X_k & 0 & \alpha I & 0 \\ R^{\frac{1}{2}} Y_k & 0 & 0 & \alpha I \end{bmatrix} \geq 0, \quad (15)$$

$$\begin{bmatrix} 1 & c_j X_k + d_j Y_k \\ X_k c_j^T + Y_k^T d_j^T & X_k \end{bmatrix} \geq 0, \quad (16)$$

$$X_0 = X_N, \quad j = 1, \dots, p,$$

are satisfied for $k = 0, \dots, N - 1$. Then with

$$P_k = X_k^{-1} \alpha \quad (17)$$

the following holds:

- a.) The optimal control problem (4) is feasible at all future time instants $k > 0$.
- b.) The closed-loop of the MPC scheme defined by (3)-(7) is asymptotically stable, while input and state constraints (2) are satisfied.

Proof 3. Let the periodically time-varying feedback matrix defining the terminal controller (10) be

$$K_{k+N} = K_k = Y_k X_k^{-1}. \quad (18)$$

Substituting X_k and Y_k in (15) by P_k^{-1} and K_k according to (17) and (18), applying the Schur complement, and pre- and post-multiplying with P_k we obtain that

$$\begin{aligned} (A_k^T + K_k^T B_k^T) P_{k+1} (A_k + B_k K_k) \\ - P_k + Q + K_k^T R K_k \leq 0 \end{aligned} \quad (19)$$

holds for $k = 0, \dots, N - 1$. We can conclude from the periodicity of the system matrices A_k and B_k that if we apply the matrices P_k and K_k periodically, i.e. $P_{k+N} = P_k$ and $K_{k+N} = K_k$, inequality (19) is satisfied for all $k \geq 0$. With the terminal control law (10) and the system dynamics (1) we have that

$$x_{k+1}^T P_{k+1} x_{k+1} + x_k^T (Q + K_k^T R K_k - P_k) x_k \leq 0 \quad (20)$$

holds for all $k \geq 0$. It clearly follows that

$$x_{k+1}^T P_{k+1} x_{k+1} < x_k^T P_k x_k, \quad \forall k \geq 0. \quad (21)$$

Thus, if for any $k \geq 0$ the state x_k lies in the ellipsoid

$$\mathcal{E}_k = \{x \in \mathbb{R}^n : x^T P_k x \leq \alpha\} \quad (22)$$

the state x_{k+1} lies in the ellipsoid

$$\mathcal{E}_{k+1} = \{x \in \mathbb{R}^n : x^T P_{k+1} x \leq \alpha\}. \quad (23)$$

Substituting X_k and Y_k in (16) by P_k^{-1} and K_k , applying the Schur complement, pre- and post-multiplying with P_k , and using the periodicity of P_k and K_k it can be shown that satisfaction of (16) implies satisfaction of (12) for all $k \geq 0$. Using this it follows from Lemma 1 that $\mathcal{E}_k \subset \mathcal{C}_k$ for all $k \geq 0$. Thus, if at initial time $k = 0$ the state x_0 is contained in the ellipsoid \mathcal{E}_0 , which lies in the constraint set \mathcal{C}_0 , the control law (10) assures satisfaction of input and state constraints for all $k \geq 0$.

Since the solution to the optimal control problem (4) enforces $x_{k+H|k} \in \mathcal{E}_{k+H}$ from these considerations in particular follows that $x_{k+H|k} \in \mathcal{C}_{k+H}$. Therefore, the terminal control law $\tilde{u}_{k+H|k} = K_{k+H} x_{k+H|k}$ satisfies the input constraints and the state $x_{k+H+1|k}$ lies in the ellipsoid $\mathcal{E}_{k+H+1} \subset \mathcal{C}_{k+H+1}$. Hence, the input sequence

$$\tilde{\mathbf{u}}_{k+1} = [u_{k+1|k}^*, \dots, u_{k+H-1|k}^*, K_{k+H} x_{k+H|k}] \quad (24)$$

is a feasible, however generally suboptimal, solution to the optimization problem (4) at time $k + 1$, leading to the suboptimal cost $J_{k+1} = J(x_{k+1}, k + 1, \tilde{\mathbf{u}}_{k+1})$. By induction it follows that initial feasibility of the optimization problem (4) implies feasibility at all future time instants, which concludes the proof of property a.) in the theorem.

To prove property b.) we consider the difference of the suboptimal cost J_{k+1} at time $k+1$ and the optimal cost value $J^*(x_k, k)$ at time k , which using the definitions of u_k in (5) and \tilde{u}_{k+1} in (24) is

$$J_{k+1} - J_k^* = -x_{k|k}^T Q x_{k|k} - u_{k|k}^{*T} R u_{k|k}^* + x_{k+H|k}^T (Q + K_{k+H}^T R K_{k+H} - P_{k+H}) x_{k+H|k} + x_{k+H+1|k} P_{k+H+1} x_{k+H+1|k}. \quad (25)$$

From (20) we know in particular that the inequality

$$x_{k+H+1|k}^T P_{k+H+1} x_{k+H+1|k} + x_{k+H|k}^T (Q + K_{k+H}^T R K_{k+H} - P_{k+H}) x_{k+H|k} \leq 0 \quad (26)$$

is satisfied. It follows that

$$J_{k+1} - J_k^* \leq -x_{k|k}^T Q x_{k|k} - u_{k|k}^{*T} R u_{k|k}^*. \quad (27)$$

Using (4b), (7) and the optimality condition $J_{k+1}^* \leq J_{k+1}$ we obtain

$$J_{k+1}^* - J_k^* \leq -x_k^T Q x_k - u_k^T R u_k. \quad (28)$$

Since $J^*(x_k, k)$ is a positive definite function, see Lemma 2, system (1) is asymptotically stable (Marquez [2003]) under the MPC controller defined in (3)-(7).

Often it is desired to maximize the volume of the terminal region in order to maximize the feasible region of the considered MPC scheme, see e.g. Chen and Ballance [1999], Böhm et al. [2008], Yu et al. [2009]. Since we consider time-varying ellipsoids as terminal region, one has to decide which ellipsoid's volume should be maximized. Since initial feasibility implies feasibility at all future time instants it is reasonable to maximize the initial ellipsoid \mathcal{E}_{k+H} , which is achieved by solving the optimization problem

$$\underset{\alpha, X_0, \dots, X_{N-1}, Y_1, \dots, Y_{N-1}}{\text{minimize}} \quad -\log \det(X_{k+H}) \quad (29)$$

subject to the LMIs (15) and (16). In the case that the prediction horizon H is larger than the time period N , one has to exploit the periodicity of the matrices X_k according to (17) and (8). It is shown in Boyd et al. [1994] that (29) is a convex optimization problem.

In Lee et al. [2005] an approach based on similar LMI conditions as in Theorem 1 is derived which also allows for the calculation of periodically time-varying terminal regions. However, the approach is developed for linear parameter-varying systems where the dynamics are approximated with a PLDI. Thus, applying Lee et al. [2005] to periodically time-varying systems would require to consider the time-varying system matrices as extreme matrices of a PLDI formulation. This increases the number of LMIs to be solved significantly and introduces unnecessary conservativeness when compared to the approach presented here, possibly leading to an infeasible LMI problem. To apply Lee et al. [2005] to periodic systems would require to replace the index k of the matrices A_k and B_k in the LMIs (15) and (16) by a new index i . The LMIs then would have to hold for all $k = 0, \dots, N-1$ and all $i = 0, \dots, N-1$ instead of only for all $k = 0, \dots, N-1$ as in this paper. The approach in Yu et al. [2009] would suffer from similar problems, since as in Lee et al. [2005] the periodic system

matrices would have to be covered in a conservative PLDI formulation. Thus, the approach presented in this paper reduces conservativeness compared to existing approaches by exploiting explicitly the periodic system dynamics.

The results obtained in Theorem 1 allow the calculation of a time-varying terminal region and a time-varying terminal penalty term for the class of linear periodically time-varying systems as introduced in (1). However, often the dynamics of the considered control system are nonlinear. Therefore, in the next section we extend the results obtained in Theorem 1 to cover a more general class of systems.

4. EXTENSION TO NONLINEAR SYSTEMS

In practical control problems one often has to deal with nonlinearities in the system description. This motivates us to extend the results of the previous section to periodically time-varying systems of the form

$$x_{k+1} = f_k(x_k, u_k), \quad (30)$$

where $f_{k+N} = f_k$ describes the periodically time-varying nonlinear dynamics. To apply LMI techniques as in Section 3 we approximate the nonlinear system (30) by a polytopic linear differential inclusion (Boyd et al. [1994]). First, we assume that the dynamics of (30) can be expressed by a linear parameter-varying system

$$x_{k+1} = A_k(\theta_k)x_k + B_k(\theta_k)u_k. \quad (31)$$

The system matrices $A_k(\theta_k)$ and $B_k(\theta_k)$ depend on the time-varying parameter vector $\theta_k = [\theta_{1,k}, \theta_{2,k}, \dots, \theta_{q,k}] \in \mathbb{R}^q$, which belongs to a polytope \mathcal{P} defined by

$$\sum_{i=1}^q \theta_{i,k} = 1, \quad 0 \leq \theta_{i,k} \leq 1. \quad (32)$$

Clearly, as θ_k varies inside the polytope \mathcal{P} , the matrices of system (31) vary inside a corresponding polytope Ω_k

$$\{A_k(\theta_k) \ B_k(\theta_k)\} \in \Omega_k, \quad (33)$$

which is defined by the convex hull of q local extreme matrices $[A_{i,k} \ B_{i,k}]$, $i = 1, \dots, q$,

$$\Omega_k := \text{Co}\{[A_{1,k} \ B_{1,k}], [A_{2,k} \ B_{2,k}], \dots, [A_{q,k} \ B_{q,k}]\}. \quad (34)$$

Therefore, we can write the matrices of system (31) as

$$A_k(\theta_k) = \sum_{i=1}^q \theta_{i,k} A_{i,k}, \quad B_k(\theta_k) = \sum_{i=1}^q \theta_{i,k} B_{i,k}. \quad (35)$$

Note that the polytope Ω_k is chosen to be time-varying, which captures the periodic time-variance of the nonlinear function $f_k = f_{k+N}$, i.e. for N nonlinear functions f_k one has to find N convex hull representations as described above. This is an easier task than finding a single time-invariant convex hull representation for all functions f_0, \dots, f_{N-1} , and the obtained result is less conservative.

Remark 1. It is not naturally guaranteed that for each function f_k the corresponding convex hull representation requires the same number q of extreme matrices. However, for simplicity of notation we assume the same number

of extreme matrices for each of the N convex hull representations. The problem of different numbers of extreme matrices could be simply overcome by introducing (unnecessary) matrices lying in the already defined convex hull, which would lead to LMI conditions that would never be active and therefore would not lead to more conservative results. Therefore, the assumption made in this paper is not limiting.

The following theorem provides LMI conditions for the calculation of a terminal region and a terminal penalty term such that the MPC scheme defined by (3)-(7) is asymptotically stable, where (4b) has to be replaced by the nonlinear dynamics (30).

Theorem 2. Suppose that the open-loop optimal control problem (4) has a feasible solution at initial time $k = 0$, where (4b) is suitably replaced by (30). Further suppose that there exist matrices $0 < X_k = X_k^T \in \mathbb{R}^{n \times n}$ and $Y_k \in \mathbb{R}^{m \times n}$, and a constant $\alpha \in \mathbb{R}^+$ such that the LMIs

$$\begin{bmatrix} X_k & X_k A_{i,k}^T + Y_k^T B_{i,k}^T & X_k Q^{\frac{1}{2}} & Y_k^T R^{\frac{1}{2}} \\ A_{i,k} X_k + B_{i,k} Y_k & X_{k+1} & 0 & 0 \\ Q^{\frac{1}{2}} X_k & 0 & \alpha I & 0 \\ R^{\frac{1}{2}} Y_k & 0 & 0 & \alpha I \end{bmatrix} \geq 0, \quad (36)$$

$$\begin{bmatrix} 1 & c_j X_k + d_j Y_k \\ X_k c_j^T + Y_k^T d_j^T & X_k \end{bmatrix} \geq 0, \quad (37)$$

$$X_0 = X_N, \quad j = 1, \dots, p, \quad i = 1, \dots, q,$$

are satisfied for $k = 0, \dots, N - 1$. Then with $P_k = X_k^{-1} \alpha$ the following holds:

- The optimal control problem (4) is feasible at all future time instants $k > 0$.
- The closed-loop of the MPC scheme defined by (3)-(7) is asymptotically stable, while input and state constraints (2) are satisfied.

Proof 4. The proof goes along the lines of the proof of Theorem 1. Substituting X_k and Y_k in (36) as in Theorem 1 by the periodic matrices P_k^{-1} and K_k , applying the Schur complement and pre- and post-multiplying with P_k we have

$$\begin{aligned} & (A_{i,k}^T + K_k^T B_{i,k}^T) P_{k+1} (A_{i,k} + B_{i,k} K_k) \\ & - P_k + Q + K_k^T R K_k \leq 0, \end{aligned} \quad (38)$$

for $i = 1, \dots, q$, and for all k . Since $\theta_{i,k} \geq 0$ this implies that the inequality

$$\begin{aligned} & \sum_{i=1}^q \theta_{i,k} ((A_{i,k}^T + K_k^T B_{i,k}^T) P_{k+1} (A_{i,k} + B_{i,k} K_k) \\ & - P_k + Q + K_k^T R K_k) \leq 0 \end{aligned} \quad (39)$$

holds for all k . Using (32) and the definition of $A_k(\theta_k)$ and $B_k(\theta_k)$ in (35) this results in

$$\begin{aligned} & (A_k^T(\theta_k) + K_k^T B_k^T(\theta_k)) P_{k+1} (A_k(\theta_k) + B_k(\theta_k) K_k) \\ & - P_k + Q + K_k^T R K_k \leq 0. \end{aligned}$$

With the terminal control law (10) and the system dynamics (31) we finally obtain that

$$x_{k+1}^T P_{k+1} x_{k+1} + x_k^T (Q + K_k^T R K_k - P_k) x_k \leq 0 \quad (40)$$

holds for all $k \geq 0$. This inequality is identical to (20) in the proof of Theorem 1. Therefore, from this point the proof can be continued in exactly the same way.

For simplicity, in the following section we provide simulation results of an example system with linear periodic dynamics as considered in Section 3. However, as illustrated in Reble et al. [2009] the application of the nonlinear extensions presented in this section would be straightforward.

5. SIMULATION RESULTS

To illustrate the results derived in Section 3 we consider an example system of the form (1) with time period $N = 3$. The system is of third order and has two inputs u_k^1 and u_k^2 , i.e. $n = 3$ and $m = 2$. It is defined by the matrices

$$\begin{aligned} A_0 &= \begin{bmatrix} 0.4 & 0.3 & 1.0 \\ 0.6 & 0.6 & 0.6 \\ 0.1 & 0.9 & 0.1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0.3 & 0.3 \\ 0.5 & 1.0 \\ 0.4 & 1.0 \end{bmatrix}, \\ A_1 &= \begin{bmatrix} 0.4 & 0.7 & 0.8 \\ 0.1 & 1.0 & 0.1 \\ 0.3 & 0.4 & 0.8 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.2 & 0.6 \\ 0.1 & 0.1 \\ 0.3 & 0.9 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} 0.3 & 0.7 & 0.3 \\ 0.8 & 0.5 & 0.3 \\ 0.6 & 0.4 & 0.7 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.9 & 0.0 \\ 0.5 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}. \end{aligned} \quad (41)$$

The initial condition for the system states is given by $x_0 = [10 \ 10 \ 5]^T$. For simplicity, we consider only input constraints of the form $-3 \leq u_k^1 \leq 3$ and $-4.5 \leq u_k^2 \leq 4.5$ for all $k \geq 0$. Thus, the number of constraints is $p = 4$. Since only input constraints are considered, the state constraint vectors are $c_j = [0 \ 0 \ 0] \ \forall j = 1, \dots, 4$. The input constraint vectors are $d_1 = -d_2 = [\frac{1}{3} \ 0]$ and $d_3 = -d_4 = [0 \ \frac{2}{9}]$. As design parameters for the predictive controller we have chosen

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}. \quad (42)$$

Solving the LMIs according to Section 3 we obtain the matrices

$$\begin{aligned} P_0 &= \begin{bmatrix} 1.87 & 0.97 & 0.52 \\ 0.97 & 2.92 & 0.65 \\ 0.52 & 0.65 & 2.13 \end{bmatrix}, \quad K_0^T = - \begin{bmatrix} 0.15 & 0.28 \\ 0.24 & 0.49 \\ 0.10 & 0.14 \end{bmatrix}, \\ P_1 &= \begin{bmatrix} 1.64 & 1.61 & 1.32 \\ 1.61 & 6.00 & 3.18 \\ 1.32 & 3.18 & 3.78 \end{bmatrix}, \quad K_1^T = - \begin{bmatrix} 0.09 & 0.24 \\ 0.23 & 0.56 \\ 0.18 & 0.50 \end{bmatrix}, \\ P_2 &= \begin{bmatrix} 3.27 & 1.78 & 1.45 \\ 1.78 & 2.67 & 1.21 \\ 1.45 & 1.21 & 2.17 \end{bmatrix}, \quad K_2^T = - \begin{bmatrix} 0.36 & 0.23 \\ 0.36 & 0.17 \\ 0.24 & 0.21 \end{bmatrix}, \end{aligned}$$

which define the time-varying terminal region, terminal penalty term and terminal controller. Using these matrices in the MPC scheme (3)-(7) leads to the simulation results shown in Figure 1. The figure exemplarily shows the states x_k^1 and x_k^3 as well as the input trajectories u_k^1 and u_k^2 , and illustrates well the effectiveness of the proposed model predictive control scheme.

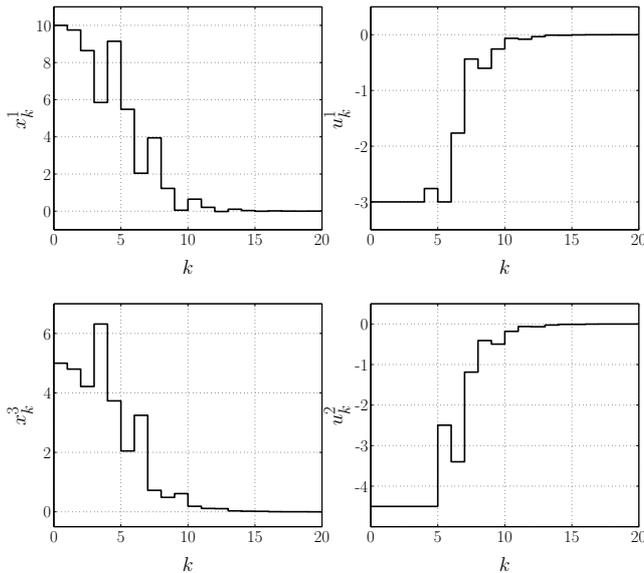


Fig. 1. Simulation results of the proposed model predictive control scheme.

6. CONCLUSIONS

In this paper we derived an LMI based approach to calculate stabilizing terms of a model predictive control scheme for the class of periodically time-varying systems. The solution to the obtained LMI conditions delivers periodically time-varying terminal regions and terminal penalty terms which render the closed-loop system asymptotically stable. We showed that the results presented improve existing approaches since we explicitly consider the periodic nature of the system dynamics. A simulation example illustrated the applicability of the proposed approach, however further research is necessary to apply the results to practical problems as e.g. to satellite control problems (Psiaki [2001], Bittanti and Colaneri [2009]).

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